

# PROOF OF A CONJECTURE OF SUN

HAO PAN

ABSTRACT. We confirm a conjecture of Sun.

Recently, Z.-W. Sun [1] proved that for any  $k \geq 1$ ,

$$\frac{1}{(2^k - 2)n + 1} \binom{(2^k - 1)n}{n} \binom{2(2^k - 1)n}{(2^k - 1)n}$$

is divisible by

$$2^{k-1} \binom{2n}{n}.$$

One key of Sun's proof is the following lemma:

*For positive integers  $n$  and  $k$ , the number of 1's in the binary expansion of  $(2^k - 1)n$  is at least  $k$ .*

In fact, Sun got a stronger result:

*For a prime  $p$  and positive integers  $n$  and  $k$ , The sum of all digits in the expansion of  $(p^k - 1)n$  in base  $p$  is at least  $k(p - 1)$ .*

Motivated by the above results, Sun made the following conjecture.

**Conjecture 1.** (I) *Suppose that  $n, m, k$  are positive integers and  $m \geq 2$ . Then there are at least  $k$  non-zero digits in the expansion of  $\frac{m^k - 1}{m - 1}n$  in base  $m$ .*

(II) *Suppose that  $n, m, k$  are positive integers and  $m \geq 2$ . Then the sum of all digits in the expansion of  $(m^k - 1)n$  in base  $m$  is at least  $k(m - 1)$ .*

In this short note, we shall confirm Conjectures 1.

**Theorem 1.** *Suppose that  $m \geq 2$ ,  $k \geq 1$  and  $a_1, \dots, a_k \in \mathbb{N}$  are not all zero. Let  $d$  be a divisor of  $m^k - 1$ . Suppose that  $\tau(x_1, \dots, x_n)$  is a nonnegative integer-valued symmetric function satisfying that*

$$\tau(mq + b, x_2, \dots, x_{k-1}, x_k) \geq \tau(b, x_2, \dots, x_{k-1}, x_k + q)$$

*for any  $q \geq 1$  and  $0 \leq b < m$ . If*

$$a_1 m^{k-1} + a_2 m^{k-2} + \dots + a_{k-1} m + a_k \equiv 0 \pmod{d},$$

*then*

$$\tau(a_1, a_2, \dots, a_n) \geq \min_{1 \leq t \leq (m^k - 1)/d} \{\tau^\circ(td)\},$$

where

$$\tau^\circ(h) = \tau(c_1, c_2, \dots, c_k)$$

if  $0 \leq h < m^k$  has an  $m$ -adic expansion  $h = c_1 m^{k-1} + c_2 m^{k-2} + \dots + c_k$ .

*Proof.* Let

$$S = \{(a_1, \dots, a_k) : a_1, \dots, a_k \in \mathbb{N} \text{ are not all zero, } \sum_{j=1}^k a_j m^{k-j} \equiv 0 \pmod{d}\}.$$

Since

$$m \left( \sum_{j=1}^k a_j m^{k-j} \right) = \sum_{j=1}^k a_j m^{k-j+1} \equiv a_1 + \sum_{j=1}^{k-1} a_{j+1} m^{k-j} \pmod{d},$$

$(a_1, a_2, \dots, a_k) \in S$  implies that  $(a_2, a_3, \dots, a_k, a_1) \in S$ . For  $\mathbf{x} = (a_1, \dots, a_k)$ , define

$$\sigma(\mathbf{x}) = a_1 + \dots + a_k$$

Let

$$S^* = \{(a_1, \dots, a_k) \in S : \tau(a_1, \dots, a_k) = \min_{(a_1, \dots, a_k) \in S} \{\tau(a_1, \dots, a_k)\}\}$$

Choose an  $\mathbf{x} = (a_1, \dots, a_k) \in S^*$  such that

$$\sigma(\mathbf{x}) = \min_{(a_1, \dots, a_k) \in S^*} \{\sigma(a_1, \dots, a_k)\}.$$

And noting that  $\tau$  is symmetric, without loss of generality, we may assume that  $a_1 \geq \max\{a_2, a_3, \dots, a_k\}$ . We shall prove that  $a_1 < m$ . Assume on the contrary that  $a_1 \geq m$ . Write  $a_1 = mq + b$  with  $0 \leq b < m$  and  $q \geq 1$ . Then

$$a_1 m^{k-1} = (mq + b)m^{k-1} \equiv bm^{k-1} + q \pmod{d}.$$

Hence  $\mathbf{x}^* = (b, a_2, \dots, a_{k-1}, a_k + q) \in S$ . Note that now

$$\tau(a_1, \dots, a_k) = \tau(mq + b, a_2, \dots, a_{k-1}, a_k) \geq \tau(b, a_2, \dots, a_{k-1}, a_k + q)$$

Hence  $\mathbf{x}^*$  also lies in  $S^*$ . But clearly

$$\sigma(\mathbf{x}) - \sigma(\mathbf{x}^*) = a_1 + a_k - (b + a_k + q) = (m-1)q \geq 1,$$

i.e.,  $\sigma(\mathbf{x}^*) < \sigma(\mathbf{x})$ . This evidently leads to a contradiction with the choice of  $\mathbf{x}$ .

So we must have  $a_1 < m$ , i.e.,  $\max\{a_1, a_2, \dots, a_k\} \leq m-1$ . Thus  $a_1 m^{k-1} + \dots + a_k = t_0 d$  with  $1 \leq t_0 \leq (m^k - 1)/d$ , and

$$\tau(a_1, a_2, \dots, a_n) = \tau^\circ(t_0 d) \geq \min_{1 \leq t \leq (m^k - 1)/d} \{\tau^\circ(td)\}.$$

□

**Corollary 1.** *Suppose that  $m \geq 2$ ,  $k \geq 1$  and  $a_1, \dots, a_k \in \mathbb{N}$  are not all zero. If*

$$a_1 m^{k-1} + a_2 m^{k-2} + \dots + a_{k-1} m + a_k \equiv 0 \pmod{\frac{m^k - 1}{m - 1}},$$

*then*

$$\sum_{j=1}^k \left\lceil \frac{a_j}{m} \right\rceil \geq k,$$

*where  $\lceil x \rceil = \min\{z \in \mathbb{Z} : z \geq x\}$ .*

*Proof.* Let

$$\tau(x_1, \dots, x_k) = \sum_{j=1}^k \left\lceil \frac{x_j}{m} \right\rceil.$$

Note that

$$\left\lceil \frac{mq + b}{m} \right\rceil = q + \left\lceil \frac{q + b}{m} \right\rceil \geq q + \left\lceil \frac{b}{m} \right\rceil,$$

and

$$\left\lceil \frac{x_k + q}{m} \right\rceil \leq \left\lceil \frac{x_k}{m} \right\rceil + \left\lceil \frac{q}{m} \right\rceil \leq \left\lceil \frac{x_k}{m} \right\rceil + q.$$

We have  $\tau(mq + b, x_2, \dots, x_{k-1}, x_k) \geq \tau(b, x_2, \dots, x_{k-1}, x_k + q)$ . Hence  $\tau$  satisfies the requirements of Theorem 1. Thus by Theomre 1,

$$\begin{aligned} \sum_{j=1}^k \left\lceil \frac{a_j}{m} \right\rceil &= \tau(a_1, \dots, a_k) \geq \min_{1 \leq t \leq m-1} \left\{ \tau^\circ \left( t \frac{m^k - 1}{m - 1} \right) \right\} \\ &\geq \min_{1 \leq t \leq m-1} \left\{ \tau^\circ \left( \sum_{j=1}^k t m^{k-j} \right) \right\} = \min_{1 \leq t \leq m-1} \left\{ \sum_{j=1}^k \left\lceil \frac{t}{m} \right\rceil \right\} = k. \end{aligned}$$

□

Let us explain why Corollary 1 implies Part (I) of Conjecture 1. White

$$n \cdot \frac{m^k - 1}{m - 1} = b_1 m^h + b_2 m^{h-1} + \dots + b_{h-1} m + b_h$$

with  $0 \leq b_i < m$ . Let

$$a_j = \sum_{\substack{1 \leq i \leq h \\ h+1-i \equiv k+1-j \pmod{k}}} b_i.$$

Since all  $b_i$  is less than  $m$ , we have

$$\left\lceil \frac{a_j}{m - 1} \right\rceil \leq |\{1 \leq i \leq h : h + 1 - i \equiv k + 1 - j \pmod{k}, b_i > 0\}|.$$

On the other hand,

$$\sum_{i=1}^h b_i m^{h+1-i} \equiv \sum_{j=1}^k a_j m^{k+1-j} \equiv 0 \pmod{(m^k - 1)/(m - 1)}.$$

It follows from Theorem 1 that

$$|\{1 \leq i \leq h : b_i > 0\}| \geq \sum_{j=1}^k \left\lceil \frac{a_j}{m-1} \right\rceil \geq \sum_{j=1}^k \left\lceil \frac{a_j}{m} \right\rceil \geq k.$$

Furthermore, Part (II) of Conjecture 1 is an immediate consequence of the following corollary.

**Corollary 2.** *Suppose that  $m \geq 2$ ,  $k \geq 1$  and  $a_1, \dots, a_k \in \mathbb{N}$  are not all zero. If*

$$a_1 m^{k-1} + a_2 m^{k-2} + \dots + a_{k-1} m + a_k \equiv 0 \pmod{m^k - 1},$$

*then*

$$\sum_{j=1}^k \left\lfloor \frac{a_j}{m-1} \right\rfloor \geq k,$$

where  $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$ .

*Proof.* Since the case  $m = 2$  easily follows from Corollary 1, we may assume that  $m \geq 3$ . Let

$$\tau(x_1, \dots, x_k) = \sum_{j=1}^k \left\lfloor \frac{x_j}{m-1} \right\rfloor.$$

Since

$$\left\lfloor \frac{mq + b}{m-1} \right\rfloor \geq q + \left\lfloor \frac{b}{m-1} \right\rfloor,$$

and

$$\left\lfloor \frac{x_k + q}{m-1} \right\rfloor \leq \left\lfloor \frac{x_k}{m-1} \right\rfloor + \left\lfloor \frac{q}{m-1} \right\rfloor + 1 \leq \left\lfloor \frac{x_k}{m-1} \right\rfloor + q,$$

we have  $\tau(mq + b, x_2, \dots, x_{k-1}, x_k) \geq \tau(b, x_2, \dots, x_{k-1}, x_k + q)$ . Thus  $\tau$  satisfies the requirements of Theorem 1. And applying Theorem 1, we get

$$\sum_{j=1}^k \left\lfloor \frac{a_j}{m-1} \right\rfloor \geq \tau^\circ(m^k - 1) = \tau^\circ\left(\sum_{j=1}^k (m-1) \cdot m^{k-j}\right) = \sum_{j=1}^k \left\lfloor \frac{m-1}{m-1} \right\rfloor = k.$$

□

**Acknowledgment.** I am grateful to Professor Zhi-Wei Sun for his very helpful suggestions on this paper.

## REFERENCES

- [1] Z.-W. Sun, *On divisibility concerning binomial coefficients*, preprint, arXiv:1005.1054.  
*E-mail address:* `haopan79@yahoo.com.cn`

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, PEOPLE'S REPUBLIC OF CHINA